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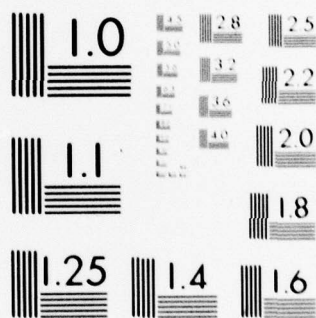
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A STUDY OF THE SINGLY TRUNCATED NORMAL DISTRIBUTION  
Final Report

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The final report for the Office of Naval Research Contract N00014-77-C-0624, A Study of the Singly Truncated Normal Distribution, consists of three papers. "Theory and Application of the Truncated Normal Distribution" was an invited paper at the Workshop on Interpretation of Remotely Sensed Data sponsored by the Institute for Atmospheric Optics and Remote Sensing. The paper will appear in the proceedings which will be published under the title Remote Sensing of the Atmosphere and Ocean by Academic Press. The second paper, "A Sequential Procedure for Estimating Clear Radiance," will appear in the preprint volume of the Sixth Conference on Probability and Statistics in Atmospheric Science published by the American Meteorological Society. "Estimating Cloud Parameters Using Truncated Distributions" was given at the COSPAR Symposium on Remote Sounding of the Atmosphere from Space. An abstract of this paper appeared in the Proceedings of the Symposium published by Pergamon Press.

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THEORY AND APPLICATION OF THE TRUNCATED NORMAL  
DISTRIBUTION FOR REMOTELY SENSED DATA<sup>1</sup>

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I. INTRODUCTION

A fundamental problem in measuring the infrared radiation emitted from the surface or lower atmosphere of the earth is the presence of clouds. The usual effect of clouds in an instrument field-of-view is to lower the measured radiation. In order to be able to use many of the techniques of remote sensing it is necessary to have an estimate of the emitted radiance from a clear or cloud-free field-of-view. The technique which is developed in this paper uses the measured radiances from a large number of neighboring small fields-of-view. If a sufficient number of these small fields-of-view are cloud-free, then a technique which uses the theory of the truncated normal distribution can be applied to the data to obtain an estimate of the clear radiance (Crosby and Glasser, 1978).

<sup>1</sup>Part of this paper was written while Dr. Crosby was supported by the Office of Naval Research Contract N00014-77C-0624.

Since the technique developed is a maximum likelihood procedure, standard statistical procedures can be used to determine if the distribution in the tail is approximately normal. This allows for a simple decision procedure to check if the model is satisfied (DePriest 1976, Crosby and DePriest 1977). These decision procedures use classical statistical goodness-of-fit tests. Finally, one difficulty with the application of this procedure has been an objective method of determining a truncation point. A sequential procedure for determining this truncation point is outlined.

## II. THE MODEL

In order to apply the technique of the truncated normal the following conditions must be satisfied. (1) There are a number of neighboring measurements where the radiances in the absence of clouds would be the same. (2) The presence of clouds lowers the radiance in the field-of-view. (3) Some of the fields-of-view are cloud-free. (4) The difference between the measured radiances from two cloud-free fields-of-view is due only to instrument noise, whose distribution is assumed to be normal.

It is assumed that the density of the measured radiances from all the fields-of-view in a set takes the form



$$h(x) = \alpha f(x) + (1 - \alpha) g(x). \quad (1)$$

In equation 1,  $\alpha$  is equal to the proportion of fields-of-view which are cloud-free,

$$f(x) = (1/2\pi\sigma^2)^{1/2} \exp(-(x-\mu)^2/2\sigma^2), \quad (2)$$

and  $g(x)$  is some unknown density function. It is further assumed that there is some  $T$  such that

$$g(x) = 0 \quad x > T. \quad (3)$$

That is for the values of  $x$  above  $T$ , the density curve has the form of a normal density curve with parameters  $\mu$  and  $\sigma^2$ . The parameter  $\mu$  is the true clear radiance. That is  $\mu$  is the radiance that would be measured from a clear field-of-view if there were no instrument noise. The variance,  $\sigma^2$ , of the instrument noise is assumed to be known. The problem is to obtain a good estimate of  $\mu$ .

### III. THE ESTIMATION PROCEDURE

The statistical technique outlined in this section is well known and is available in the literature. See, for example, Crosby and Glasser 1978.

If the radiance data above some truncation point  $T$  is from a truncated normal distribution, that is it has the form given by equation 2, and if it has known variance  $\sigma^2$ , then the maximum likelihood estimate for the mean  $\mu$  is a monotonic function of the arithmetic mean of the

observations above that truncation point.

Let  $I_{\lambda i}$  ( $i = 1, \dots, n$ ) be the measured radiance values above some truncation point  $T$ . Let

$$Z_i = (I_{\lambda i} - T)/\sigma \quad (4)$$

and let

$$\hat{v} = (T - \hat{u})/\sigma. \quad (5)$$

Recall that  $T$  is a known truncation point,  $\sigma$  is the known instrument noise, and  $\hat{u}$  is the estimate of the clear radiance. Let

$$\bar{Z} = \sum_{i=1}^n Z_i \quad (6)$$

then  $\hat{v}$  is given by the solution to the equation

$$-\hat{v} + (\exp(-\hat{v}^2/2) / \int_{\hat{v}}^{\infty} \exp(-x^2/2) dx) = \bar{Z}. \quad (7)$$

Once  $\hat{v}$  is known,  $\hat{u}$  is found by the inverse of equation (5)

$$\hat{u} = T - \hat{v} \sigma. \quad (8)$$

Since the function given in equation 8 is monotonic and smooth, so is its inverse. It is possible to find a good rational polynomial approximation to the inverse function which gives  $\hat{v}$  as a function of  $\bar{Z}$ .



As an example this procedure has been applied to the 10.5 - 12.5  $\mu\text{m}$  infrared channel of the Scanning Radiometer of the NOAA satellites. Table 1 gives the upper tail of one such data set. The measurements are in counts and have been transformed in order to keep the example as simple as possible.

TABLE 1

Data from the 10.5 - 12.5  $\mu\text{m}$  infrared channel of the Scanning Radiometer.

Counts	Frequency	Estimated Mean
C	$f_1$	$\hat{\mu}$
73	2	
72	3	
71	8	
70	12	
69	22	
68	31	65.0

For the data in Table 1,  $T = 67.5$ ,  $\sigma = 3$  and hence  $\bar{Z} = .560$ . Then  $\hat{v} = .833$  and  $\hat{\mu} = 67.5 - ((.833)3) = 65.0$ . The value for  $\sigma$  is found from independent data and  $\hat{v}$  is found from  $\bar{Z}$  by solving equation 7.

#### IV. GOODNESS-OF-FIT TESTS

The procedures outlined in section III depend on the assumption that the upper tail of the distribution has the

shape of a truncated normal distribution. If the shape is not normal it may indicate that few, if any, of the fields-of-view are completely cloud-free. Hence the shape of the data from the tail may be used to screen out situations of this type.

The test which is recommended is the chi-square goodness-of-fit test. For a description of other possible tests see DePriest 1976. The chi-square test is well known. See, for example, Cochran 1952. However, for completeness, a short description of the procedure will be given here.

Given  $n$  observations grouped into  $K$  mutually exclusive categories let  $O_i$  and  $E_i$  ( $i = 1, 2, \dots, k$ ) denote respectively the observed and expected frequencies for the  $K$  categories. The statistic is

$$\chi^2 = \sum_{i=1}^K (O_i - E_i)^2 / E_i \quad . \quad (9)$$

Since the estimate of the parameter  $\mu$ ,  $\hat{\mu}$ , is a maximum likelihood estimate the asymptotic distribution of the statistic  $\chi^2$  will be chi-square with  $K - 2$  degrees of freedom. Since the radiance values are already grouped, this natural grouping is used in the computation of the

statistic. To illustrate the technique, two examples are provided.

For the data in Tables 2 and 3 it was assumed that  $\sigma = 3$ . This value was determined from independent measurements.

TABLE 2  
Data From the Scanning Radiometer

The data are in counts. The estimated mean uses the data in the tail above and including the corresponding count.

Counts	Frequency	Estimated Mean
C	$f_i$	$\hat{\mu}$
73	2	
72	3	
71	8	
70	12	
69	22	
68	31	
67	42	
66	34	
65	37	
64	49	65.3

For the data in Table 2 a truncation point of 63.5 was taken. This gives an estimate of  $\mu$ ,  $\hat{\mu} = 65.3$ . The number of groups was taken to be equal to 9. The data associated



with 72 and 73 counts are grouped into a single class. Then the chi-square statistic will have 7 degrees of freedom. It is found that  $\chi^2 = 4.5$ . This value indicates a good fit. The model given in section II is reasonable.

TABLE 3

Data From the Scanning Radiometer

The data are in counts. The estimated mean uses the data in the tail above and including the corresponding count.

Counts	Frequency	Estimated Mean
C	$f_i$	$\hat{\mu}$
50	1	45.4
49	9	
48	7	
47	9	
46	6	
45	8	
44	6	

For the data in Table 3, a truncation point of 43.5 was taken and  $\hat{\mu} = 45.4$ . Grouping the counts 50 and 49 into one class, there are 6 classes. The chi-square statistic has 4 degrees of freedom. The value is found to be 12.3. The fit is not very good and the model of an upper tail which is normal is rejected.

## V. DETERMINING THE TRUNCATION POINT

Given a truncation point  $T$ , the procedure outlined in the earlier sections will estimate  $\mu$  and test if the tail is normal. However one difficulty with the application of these techniques has been a determination of a good truncation point. The following procedure is recommended. It is assumed that the data is grouped. Then the count data from the instrument will be in the format represented by Table 4.

TABLE 4

Class	Class Midpoint	Class Frequency
$k_0 - k_1$	$m_1 = (k_1 + k_0)/2$	$f_1$
$k_1 - k_2$	$m_2 = (k_2 + k_1)/2$	$f_2$
.	.	.
.	.	.
.	.	.
$k_{i-1} - k_i$	$m_i = (k_i + k_{i-1})/2$	$f_i$

The minimum number of classes used in the technique should cover approximately 2 standard deviations. In the examples which have been presented  $\sigma$  was equal to 3, and a minimum of 7 classes was used. The adjusted mean of the data above and including class  $j$  is



$$\bar{z}_j = \frac{\sum_{i=1}^j (m_i - k_j) \cdot f_i}{\sigma(\sum_{i=1}^j f_i)} \quad (10)$$

Then  $\hat{v}_j$  is found using equation 7 or an approximation to this equation. If  $\hat{v}_j > 0$ , compute a  $\bar{z}_{j+1}$ . Repeat this process until  $\hat{v}_m < 0$ . The data above the point  $k_m$  is then checked using the goodness-of-fit criteria of section 4. If the data does not fit the tail of a normal distribution the process is terminated and the retrieval discarded.

At this stage of the process the truncation point is  $k_m$ . The truncation point should be selected to include as much of the data as possible. The reason for this is seen in Table 5. Table 5 gives the asymptotic standard deviation of  $((\hat{\mu} - \mu)\sqrt{N})/\sigma$  for various values of  $T$ . In this table  $N$  is equal to the total sample size if there were no truncation,  $\sigma$  is the known standard deviation, and  $\hat{\mu}$  is the estimate of  $\mu$  found using the maximum likelihood technique of section 3. It is seen from this table that there is a significant reduction in the standard error of estimate for  $\hat{\mu}$  if the truncation point changes from  $\mu$  to two standard deviations below  $\mu$ .

TABLE 5

Asymptotic Standard Deviation of  $((\hat{\mu} - \mu)\sqrt{N})/\sigma$ 

$T^*$	Asymptotic Standard Deviation
0.5	3.474
0.	2.346
-0.5	1.373
-1.5	1.178
-2.	1.074
-3.	1.007

(In Table 5  $T^* = (T - \mu)/\sigma$ .)

A sequential procedure is used to determine if the truncation point should be moved. Let  $I_{\lambda 1}, I_{\lambda 2}, \dots, I_{\lambda n_1}$  be the radiance measurements above some  $T_1$  and let  $I_{\lambda 1}, I_{\lambda 2}, \dots, I_{\lambda n_1}, \dots, I_{\lambda n_2}$  be the radiance measurements above some  $T_2$  ( $T_2 < T_1$ ). A decision must be made whether to move the truncation point to  $T_2$  and hence increase the number of measurements used to estimate  $\mu$ . The decision process is based upon the result that using the information from the values  $I_{\lambda 1}, I_{\lambda 2}, \dots, I_{\lambda n_1}$ , it is possible to estimate  $n_2$ . This estimate is called  $\hat{n}_2$ . It can also be shown that  $(\hat{n}_2 - n_2)/n_2^{\frac{1}{2}}$  is asymptotical normal with mean 0 and estimatable standard deviation. If  $(\hat{n}_2 - n_2)/n_2^{\frac{1}{2}}$  is within certain bounds then the truncation point is extended to  $T_2$ . If  $(\hat{n}_2 - n_2)/n_2^{\frac{1}{2}}$  is outside these bounds, then

$T_1$  is used as the truncation point and  $\hat{\mu}_1$ , the estimate based on the variables  $I_{\lambda 1}, I_{\lambda 2}, \dots, I_{\lambda n_1}$  is used as the estimator for  $\mu$ . The process is repeated using  $T_2$  in place of  $T_1$  and some  $T_3 (T_3 < T_2)$  in place of  $T_2$ . This procedure is repeated with a sequence of  $T_i$ 's until  $(\hat{n}_i - n_i) / n_i^{\frac{1}{2}}$  is outside of the bounds or until  $T^* = ((T - \mu)/\sigma)$  is less than -2.

Let  $\hat{\mu}_1$  be the estimator of  $\mu$  based on the measurements above  $T_1$ . Let

$$F_c(T_1, \hat{\mu}, \sigma^2) = (1/2\pi\sigma^2)^{\frac{1}{2}} \int_{T_1}^{\infty} \exp(-(x - \hat{\mu})^2/2\sigma^2) dx. \quad (11)$$

Then an estimator of  $n_2$  is

$$\hat{n}_2 = [n_1 / (F_c(T_1, \hat{\mu}_1, \sigma^2) / F_c(T_2, \hat{\mu}_1, \sigma^2))] \quad (12)$$

where  $[y]$  denotes the greatest integer less than or equal to  $y$ . Let  $n_2$  be the actual number of measurements above  $T_2$ . Assuming that  $I_{\lambda 1}, I_{\lambda 2}, \dots, I_{\lambda n_2}$  come from a population which is normal with mean  $\mu$  and variance  $\sigma^2 = 1$ , the asymptotic variance of  $(\hat{n}_2 - n_2)/n$  is given by

$$S^2 = \frac{(F_c(T_2, \mu) \Psi(T_1, \mu) - F_c(T_1, \mu) \Psi(T_2, \mu))^2}{F_c(T_2, \mu) (F_c(T_1, \mu)) H(T_1, \mu)} + \frac{F_c(T_2, \mu) - F_c(T_1, \mu)}{F_c(T_1, \mu)} \quad (13)$$

where

$$\Psi(T, \mu) = (1/2\pi)^{-\frac{1}{2}} \exp(-(T - \mu)^2/2) \quad (14)$$

$$F_c(T, \mu) = (1/2\pi)^{-\frac{1}{2}} \int_T^{\infty} \exp(-(x - \mu)^2/2) dx \quad (15)$$

$$H(T_1, \mu) = F_c^2(T_1, \mu) + (T_1 - \mu) \cdot F_c(T_1, \mu) \Psi(T_1, \mu) - \Psi(T_1, \mu)^2 .$$

Table 6 gives the asymptotic standard deviations for some values of  $T_1$  and  $T_2$  when  $\mu = 0$ .

TABLE 6

The Asymptotic Standard Deviations of  $(n_2 - n_2)/n_2^{\frac{1}{2}}$

$T_1$	$T_2$	S
1.0	.67	1.064
0.5	.17	.833
0.0	-.33	.632
-0.5	-.83	.460
-1.0	-1.33	.316
-1.5	-1.83	.202
-2.0	-2.33	.119
-2.5	-2.83	.064



This sequential procedure has been applied to data from the 10.5 - 12.5  $\mu\text{m}$  channel of the Scanning Radiometer. Table 7 gives the upper tail of such a data set. The data are in counts.

TABLE 7

Data From the Scanning Radiometer

Counts	Frequency	Estimated Mean	$(n_2 - n_2^*)/n_2^{1/2}$	S
88	1			
87	3			
86	11			
85	18	78.3		
84	31	79.0		
83	47	79.3		
82	67	79.4		
81	68	79.7		
80	71	79.9*	.57	.58
79	75	79.8	-.10	.48
78	60	79.9	.13	.37
77	57	79.8	-.52	.29
76	45	79.7	-.40	.23
75	44	79.4	-.89	.16
74	28	79.3		
73	22	79.2		
72	16	79.1		
71	26	78.8		
70	21	78.5		
69	9			
68	21			
67	20			
66	13			

\*  $\hat{v}$  is less than zero.



For this example  $\sigma = 3$ . The estimate of  $\mu$ ,  $\hat{\mu}$  uses the data in the tail above and including the corresponding counts. For example the estimate corresponding to 80 uses a truncation point of 79.5. This is the first estimate where  $\hat{v}$  is less than zero. The sequential procedure is then applied. For the other entries corresponding to 80  $T_1 = 79.5$ ,  $T_2 = 78.5$ ,  $n_2 = 317$ ,  $\hat{n}_2 = 327$  and  $n_1 = 246$ . Note that  $(\hat{n}_2 - n_2)/n_2^{\frac{1}{2}}$  is not large compared to  $S$  and hence the truncation point would be moved to  $T_2$  or 79.5. Continuing this process, if a conservative rule were used the process would terminate at  $T = 75.5$  which would yield an estimate of  $\hat{\mu} = 79.7$ . By any reasonable rule the process would terminate at  $T = 74.5$  which gives a  $\hat{\mu} = 79.4$ . Note for this last entry the value of the statistic  $(\hat{n}_2 - n_2)/n_2^{\frac{1}{2}}$  is greater than four standard deviations from 0.

## VI. CONCLUSIONS

In this paper a procedure which uses standard and new statistical techniques to estimate clear radiances for a high spacial resolution, multiple field-of-view infrared instrument has been developed. The technique does not completely solve the cloud problem. For example, it will not distinguish between a low level uniform cloud layer and the surface. Also, certain types of inversions in the

temperature profile will cause difficulty. However, the technique is computationally simple and can easily be used with other types of procedures. This technique has been applied to real data and seems to work very well.

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## A SEQUENTIAL PROCEDURE FOR ESTIMATING CLEAR RADIANCE

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### 1. INTRODUCTION

A fundamental problem in measuring the infrared radiation emitted from the surface or lower atmosphere of the earth from space is the presence of clouds. The usual effect of clouds in an instrument field-of-view is to lower the emitted radiance. In order to use many of the techniques of remote sensing it is necessary to have an estimate of the emitted radiance from a clear or cloud-free field-of-view. One technique which has been used successfully is to measure the radiation emitted from a large number of neighboring small fields-of-view. If a sufficient number of these small fields-of-view are cloud-free, then a technique which uses the theory of the truncated normal distribution can be applied to the data to obtain an estimate of the clear radiance (Crosby and Glasser, 1978). One difficulty with the application of this

technique has been determining an objective method of selecting the truncation point. In this paper a sequential procedure for determining the truncation point is developed.

## 2. THE MODEL

In order to apply the technique of the truncated normal the following conditions must be satisfied. (1) There are a number of neighboring measurements where the radiances in the absence of clouds would be the same. (2) The presence of clouds lowers the radiance in the field-of-view. (3) Some of the fields-of-view are cloud-free. (4) The difference between the measured radiances from two cloud-free fields-of-view is due only to instrument noise, whose distribution is assumed to be normal.

It is assumed that the density of the measured radiances from two cloud-free fields-of-view is due only to instrument noise, whose distribution is assumed to be normal.

It is assumed that the density of the measured radiances from all the fields-of-view in a set takes the form

$$h(x) = c \cdot f(x) + (1-c) g(x). \quad (1)$$

In equation 1,  $c$  is equal to the proportion of fields-of-view which are cloud-free,

$$f(x) = (1/2\pi\sigma^2)^{\frac{1}{2}} \exp(-(x-\mu)^2/2\sigma^2), \quad (2)$$



and  $g(x)$  is some unknown density function. It is further assumed that there is some  $T$  such that

$$g(x) = 0 \quad x > T. \quad (3)$$

That is for the values of  $x$  above  $T$ , the density curve has the form of a normal density curve with parameters  $\mu$  and  $\sigma^2$ . The parameter  $\mu$  is the true clear radiance. That is  $\mu$  is the radiance that would be measured from a clear field-of-view if there were no instrument noise. The variance,  $\sigma^2$ , of the instrument noise is assumed to be known. The problem is to obtain a good estimate of  $\mu$ . If  $T$  is known then standard results can be applied to obtain an estimate of  $\mu$ . (Crosby and Glasser, 1978). The point  $T$  is in general unknown; however, it is assumed there is a known  $T_1$  ( $T_1 > T$ ) such that  $g(x) = 0$  for  $x > T_1$ . The  $T$  in equation 3 is the optimal truncation point.

### 3. SEQUENTIAL PROCEDURE FOR ESTIMATING $T$

Let  $X_1, X_2, \dots, X_n$  be a random sample from the population with the density in equation (1). Let  $X_{11}, X_{12}, \dots, X_{1n_1}$  be the values from the sample which are above the known  $T_1$  ( $T_1$  is defined in section 2). A  $T_2$  ( $T_2 < T_1$ ) is selected. The values  $X_{11}, X_{12}, \dots, X_{1n_1}$  are used to estimate  $\mu$ . Let the maximum likelihood estimator be  $\hat{\mu}_1$ . Let

$$F_c(T_1, \hat{\mu}, \sigma^2) = (1/2\pi\sigma^2)^{\frac{1}{2}} \int_{T_1}^{\infty} \exp(-(x-\hat{\mu})/2\sigma^2) dx. \quad (4)$$

Given the values  $X_{11}, X_{12}, \dots, X_{1n_1}$  and  $\hat{\mu}_1$ , an estimator of the number of measurements  $\hat{n}_2$  above  $T_2$ , assuming that the distribution above  $T_2$  is normal, is given by

$$\hat{n}_2 = [n_1 / (F_c(T_1, \hat{\mu}_1, \sigma^2) / F_c(T_2, \hat{\mu}_1, \sigma^2))] \quad (5)$$

where  $[y]$  denotes the greatest integer less than  $y$ . Let  $n_2$  be the actual number of values above  $T_2$ . If the distribution is normal above  $T_2$  then it is possible to show that

$$(\hat{n}_2 - n_2) / (n_2)^{\frac{1}{2}}$$

is asymptotically normal, where this normal has mean 0 and known standard deviation.

If  $(\hat{n}_2 - n_2) / (n_2)^{\frac{1}{2}}$  is outside of certain bounds as determined by the asymptotic theory, then  $T_1$  is used as  $T$  and  $\hat{\mu}_1$  is used as the estimator of  $\mu$ . If  $(\hat{n}_2 - n_2) / (n_2)^{\frac{1}{2}}$  is within these bounds, then  $T_2$  is used as a truncation point and a  $T_3$  ( $T_3 < T_2$ ) is selected and the process repeated with  $T_2$  replacing  $T_1$  and  $T_3$  replacing  $T_2$ . The process is repeated until the  $(\hat{n}_i - n_i) / (n_i)^{\frac{1}{2}}$  is outside the bounds or until  $(T_i - \hat{\mu}) / \sigma$  is less than some  $C$ .

#### 4. ASYMPTOTIC THEORY

The results of this section are based on the theory given in a series of papers by Sanathanan (1972a, 1977).

Let  $X_1, X_2, \dots, X_N$  be  $N$  values of a variable  $X$  independently sampled from a distribution  $F(x)$ . Assume that two constants are given  $T_1$  and  $T_2$  ( $T_2 < T_1$ ) such that the distribution  $F(x)$  has known functional form  $F(x; \mu)$  to the right of  $T_2$ . Let  $n_2$  equal the number of variables to the right of  $T_2$ . In the argument which follows  $n_2$  is considered to be a parameter. Consider the distribution of the variables above  $T_2$ . This is a truncated distribution with density function

$$\frac{f(x; \mu)}{F_c(T_2; \mu)}$$

where  $f(x; \mu)$  is the density of  $F(x; \mu)$  and  $F_c(T_2; \mu) = 1 - F(T_2; \mu)$ . Call this distribution function  $F_{T_2}(\mu)$ . The distribution function of the variables above  $T_1$  can be considered as a truncated distribution of the distribution  $F_{T_2}(\mu)$ . The density function of this distribution will have form

$$f(x; \mu) = \frac{\frac{f(x; \mu)}{F_c(T_2; \mu)}}{\frac{F_c(T_1; \mu)}{F_c(T_2; \mu)}} = \frac{f(x; \mu)}{F_c(T_1; \mu)} \quad (6)$$



Note that  $\frac{F_c(T_1; \mu)}{F_c(T_2; \mu)} = P(\mu)$  is the conditional probability

that  $X_1$  is above  $T_1$  given that it is above  $T_2$ .

Let  $X_1, X_2, \dots, X_{n_1}$  be a random number of observations  $n_1 \leq n_2$  above  $T_1$ . The likelihood function is given by

$$L(n_2; \mu) = L_1(n_2; P(\mu)) L_2(\mu) \quad (7)$$

where

$$L_1(n_2; P(\mu)) = \frac{n_2!}{n_1! (n_2 - n_1)!} (P(\mu))^{n_1} (1 - P(\mu))^{n_2 - n_1} \quad (8)$$

and

$$L_2(\mu) = \prod_{i=1}^{n_1} g(x_i; \mu). \quad (9)$$

Only the case where  $(\hat{n}_2, \hat{\mu})$  is maximized first by obtaining  $\hat{\mu}$  as the value of  $\mu$  which maximized  $L_2(\mu)$  and then maximizing  $L_1(n_2, P(\mu))$  with respect to  $n_2$  is considered. This estimate of  $n_2$  is given by  $\hat{n}_2 = [n_1 / P[\hat{\mu}]]$  where  $[y]$  denotes the greatest integer less than  $y$ . Then as shown by Sanathanan (1977) if at every admissible value of  $\mu$  ( $\mu$  is a  $r$ -dimensional vector of parameters)  $P(\mu)$  admits continuous first-order partial derivatives and  $f(x, \mu)$  admits continuous first and second order partial derivatives almost everywhere with



respect to  $x$  then

$$\left( \frac{\hat{n}_2}{n_2}, \hat{u}, P(\hat{\mu}) \right) \rightarrow (1, \mu, P(\mu)) \text{ a.s.}$$

and

$$(n_2^{\frac{1}{2}} (\hat{u} - u); n_2^{-\frac{1}{2}} (\hat{n}_2 - n_2))$$

is asymptotically normal  $N(0, \Sigma)$  where  $\Sigma^{-1}$  is the  $(r+1) \times (r+1)$  matrix given below.

Let the partial derivatives of  $P(\mu)$  and  $\log g(x, \mu)$  with respect to  $\mu_j$  (the  $j^{\text{th}}$  component of  $\mu$ ) be denoted by  $P_j(\mu)$ ,  $g_j(x, \mu)$ . Then set

$$a_{jm} = \frac{P_j(\mu) P_m(\mu)}{P(\mu) (1 - P(\mu))} + P(\mu) \quad (10)$$

$$E(g_j(x; \mu) g_m(x; \mu)).$$

(The expected values are taken with respect to  $g(x; \mu)$ ).

$$a_{om} = \frac{P_m(\mu)}{1 - P(\mu)} \quad (11)$$

$$a_{oo} = \frac{P(\mu)}{1 - P(\mu)} \quad (12)$$

The inverse of the matrix  $\Sigma$  is given by

$$\Sigma^{-1} = \begin{bmatrix} (a_{jm}) & (a_{om}) \\ (a_{om}) & (a_{oo}) \end{bmatrix} . \quad (13)$$

For further discussion and development see the papers by Sanathanan.

If the above theory is applied to the single parameter truncated normal with variance 1, it is found that the asymptotic variance of  $(\hat{h}_2 - h_2) n^{-\frac{1}{2}}$  is given by

$$S^2 = \frac{(F_c(T_2; \mu)\Psi(T_1; \mu) - F_c(T_1; \mu)\Psi(T_2; \mu))^2}{F_c(T_2; \mu) (F_c(T_1; \mu)) H(T_1; \mu)} + \frac{F_c(T_2; \mu) - F_c(T_1; \mu)}{F_c(T_1; \mu)} \quad (14)$$

where

$$\Psi(T; \mu) = (1/2\pi)^{-\frac{1}{2}} \exp - (T - \mu)^2/2 \quad (15)$$

$$F_c(T; \mu) = (1/2\pi)^{-\frac{1}{2}} \int_T^{\infty} \exp - (x - \mu)^2/2 dx \quad (16)$$

$$H(T_1; \mu) = F_c^2(T_1; \mu) + (T_1 - \mu) \cdot F_c(T_1; \mu)\Psi(T_1; \mu) - \Psi(T_1; \mu)^2. \quad (17)$$

Table 1 gives this asymptotic variance for some values of  $T_1$  and  $T_2$ . Note that in the application  $\mu$  is estimated

and hence so is the asymptotic variance.

TABLE 1  
The Asymptotic Variance of  $(\hat{n}_2 - n_2)n_2^{-\frac{1}{2}}$

$T_1$	$T_2$	$S^2$
1.0	0.5	2.39
0.5	0.	1.33
0.	-0.5	0.700
-0.5	-1.0	0.340
-1.0	-1.5	0.148
-1.5	-2.0	0.057
-2.0	-2.5	0.019
-2.5	-3.0	0.005

## 5. EXAMPLE

The procedure has been applied to the 10.5 - 12.5  $\mu$ m infrared channel of the Scanning Radiometer on the NOAA satellites. The data appears in blocks of  $1024$  measurements. Table 2 gives the upper tail of one such data set. The measurements are in counts and have been transformed in order to keep the example as simple as possible.

For this example it is assumed  $\sigma = 3$ . This value was determined from independent measurements. It is clear from an examination of the data that a reasonable truncation point for this data is 57.5.

TABLE 2

## Data From the Scanning Radiometer

The data are in counts. The estimated mean uses the data in the tail above and including the corresponding count.

Counts	Frequency	Estimated Mean
C	$f_1$	$\mu$
73	2	
72	3	
71	8	
70	12	64.9
69	22	64.9
68	31	65.0
67	42	65.0
66	34	65.3
65	37	65.5
64	49	65.3
63	42	65.1
62	39	64.9
61	29	64.8
60	12	64.7
59	14	64.5
58	8	64.6
57	11	64.4
56	14	64.1

Let  $T_1 = 63.5$  and  $T_2 = 62.5$ . Then  $n_1 = 245$  and  $\hat{u}_1 = 65.3$ . The estimated asymptotic standard deviation of  $(\hat{n}_2 - n_2)/n_2^{1/2}$  is .45. In this case  $(\hat{n}_2 - n_2)/n_2^{1/2} = -.53$ . Move to  $T_2$  and repeat the process. Let  $T_1 = 57.5$  and  $T_2 = 56.5$ . Then  $n_1 = 393$  and  $\hat{u}_1 = 64.6$ . The estimated asymptotic standard deviation of  $(\hat{n}_2 - n_2)/n_2^{1/2}$  is .08. In this case  $(\hat{n}_2 - n_2)/n_2^{1/2} = -.44$ . Stop at  $T_1$  and use  $\mu_1 = 64.6$  as the estimate.



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## 6. CONCLUSIONS

The technique presented in this paper gives a simple objective method for finding a truncation point for a distribution which is contaminated over part of its range. For large samples with the single parameter normal it seems to work very well.

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ESTIMATING CLOUD PARAMETERS  
USING TRUNCATED DISTRIBUTIONS

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ABSTRACT

When estimating atmospheric and surface parameters from satellite measurements of infrared radiance, one of the most difficult problems is that of cloud effects. It has been shown that the theory of truncated probability distributions can be used to estimate clear radiances in certain situations. In this paper it is shown that the same techniques can be used to obtain bounds on certain cloud parameters. These parameters include cloud-top temperature and cloud amount. If the atmospheric temperature profile is assumed known, then bounds on cloud height can be obtained. These bounds can then be used to improve results for certain temperature retrieval techniques.

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### INTRODUCTION

The primary difficulty associated with remote sensing of atmospheric or surface parameters using satellite measured infrared radiation is the presence of clouds. In a previous paper, a technique which uses the theory of the truncated normal distribution to estimate cloud-free radiances has been developed. See Crosby [1975]. In this paper the technique of using the truncated normal distribution is used to obtain bounds on certain cloud parameters. These parameters include cloud-top temperature and cloud amount. If the atmospheric temperature profile is assumed known, bounds on cloud height can also be obtained.

The technique uses a large number of small fields-of-view (f-o-v's) and assumes that the noise of the instrument has approximately a normal distribution. It is also assumed that atmospheric temperature is a decreasing function of height. If these assumptions are satisfied, it is then possible to obtain an upper bound on cloud-top temperature and upper and lower bounds on cloud amount for the highest layer of clouds in a composite f-o-v.

### THE MODEL

There are assumed to be  $n$  small f-o-v's in the composite f-o-v. Let  $R_1, R_2, \dots, R_n$  be the measured radiances from these small f-o-v's. If f-o-v  $i$  has more of the higher level cloud

than f-o-v-  $j$  and if the effect of noise is neglected then  $R_i < R_j$ . That is, a f-o-v which is covered by more of the higher level cloud is assumed to have a lower true radiance. If several f-o-v's are totally cloud covered then the difference between their measured radiance values is assumed to be due to measurement error which is assumed to have approximately a normal distribution.

Let  $h(r)$  be the theoretical histogram or density from an array of radiances which meet the above conditions. Then the lower part of the histogram will have the shape of a normal curve. That is, there is a  $K$  such that if  $r < K$

$$h(r) \approx \frac{1}{\sqrt{2\pi}\sigma} \exp(-(r-u)^2/2\sigma^2) \quad (1)$$

In equation (1)  $\sigma$  is the standard deviation of the instrument noise and  $u$  is the true radiance emitted from a cloud covered f-o-v. It is assumed in the discussion that follows that  $\sigma$  is known, but this is not a necessary assumption. It will also be assumed that  $K > \mu$ . Given that the data satisfies these conditions the techniques for obtaining an estimate of  $\mu$  are well developed and have been successfully applied to the problem of estimating sea-surface temperature. See, for example, Crosby [1975]. For a discussion of the truncated normal see Johnson and Kotz [1970].

TECHNIQUE

As will be seen in the following discussion, the technique is simple to apply and could easily be carried out by a microprocessor on the satellite. In order to keep the development as simple as possible the following assumptions are made: the data is grouped or digitized; the data is in counts; and the instrument has a reverse calibration (high counts = low radiance). The data to be analyzed will be on the upper or left tail of the histogram.

The count data from the instrument will be in the format represented by Table 1.

Table 1

class	class midpoint	class frequency
$k_0 - k_1$	$m_1 = (k_1 + k_0)/2$	$f_1$
$k_1 - k_2$	$m_2 = (k_2 + k_1)/2$	$f_2$
0	0	0
0	0	0
0	0	0
$k_{i-1} - k_i$	$m_i = (k_i + k_{i-1})/2$	$f_i$

Given that the data is in this format the procedure is as follows. Select a minimum number of classes to use. A



reasonable number of classes to use as the minimum should cover approximately two standard deviations. For example, if the standard deviation is three units and each class is one unit in length, then use a minimum of six classes. Form the adjusted mean of the data above and including class  $j$ ,

$$\bar{z}_j = \frac{\sum_{i=1}^j f_i (m_i - k_j)}{\sum_{i=1}^j f_i} \quad (2)$$

Then find  $V_j$  from Table 2.

TABLE 2 V vs  $\bar{z}$

$.3 \leq \bar{z} < 1$										
$\bar{z}$	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.3	2.77	2.65	2.53	2.42	2.31	2.21	2.12	2.03	1.94	1.86
.4	1.778	1.702	1.629	1.559	1.491	1.426	1.363	1.302	1.243	1.187
.5	1.132	1.078	1.026	.976	.927	.880	.834	.789	.745	.702
.6	.661	.620	.581	.542	.504	.467	.431	.396	.361	.327
.7	.294	.262	.230	.198	.168	.138	.108	.079	.050	.022
.8	-.006	-.033	-.060	-.086	-.112	-.138	-.163	-.188	-.212	-.236
.9	-.260	-.283	-.307	-.329	-.352	-.374	-.396	-.418	-.439	-.460

$1 \leq \bar{z} < 4$

$\bar{z}$	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
1	-.481	-.678	-.856	-1.020	-1.172	-1.314	-1.449	-1.578	-1.702	-1.821
2	-1.937	-2.050	-2.161	-2.269	-2.376	-2.481	-2.586	-2.689	-2.792	-2.894
3	-2.995	-3.097	-3.198	-3.298	-3.399	-3.499	-3.599	-3.700	-3.800	-3.900

If  $V_j > 0$ , repeat, using class  $j+1$ . Continue until  $V_m < 0$ .



Then the estimate for the counts from a totally cloud covered f-o-v is given by

$$C = k_m - V_m \sigma .$$

This C can be converted to radiance or to temperature by the calibration curve of the instrument. Given the assumption that temperature decreases with height this number will be an upper bound for the cloud-top temperature. If the temperature profile is assumed known as a function of height then it is obvious how to use this temperature to obtain a bound on cloud height.

To estimate the number of f-o-v's which are totally covered by clouds the following procedure is used.

Let

$$N(k_m) = \sum_{i=1}^m f_i \quad (4)$$

where m is the number of classes used to estimate C in Eq. 3.

$N(k_m)$  is the number of measurements above  $k_m$ . Then an estimate for the number of f-o-v's which are totally covered by clouds is given by

$$W = \left( \int_{k_m}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp(-(x-c)^2/2\sigma^2) dx \right)^{-1} \cdot N(k_m) \quad (5)$$

and  $N = [W]$ , where  $[W]$  is the smallest integer greater than or equal to W. To obtain a lower bound on the proportion of the composite f-o-v covered by cloud, set

$$P_L = (N/M)$$

## Estimating Cloud Parameters Using Truncated Distributions 7

where  $M$  is the number of small f-o-v's in the composite f-o-v.

To obtain an upper bound on the proportion covered by the higher level clouds the same technique is applied to the tail of data associated with the higher radiances or lower counts.

### EXAMPLES

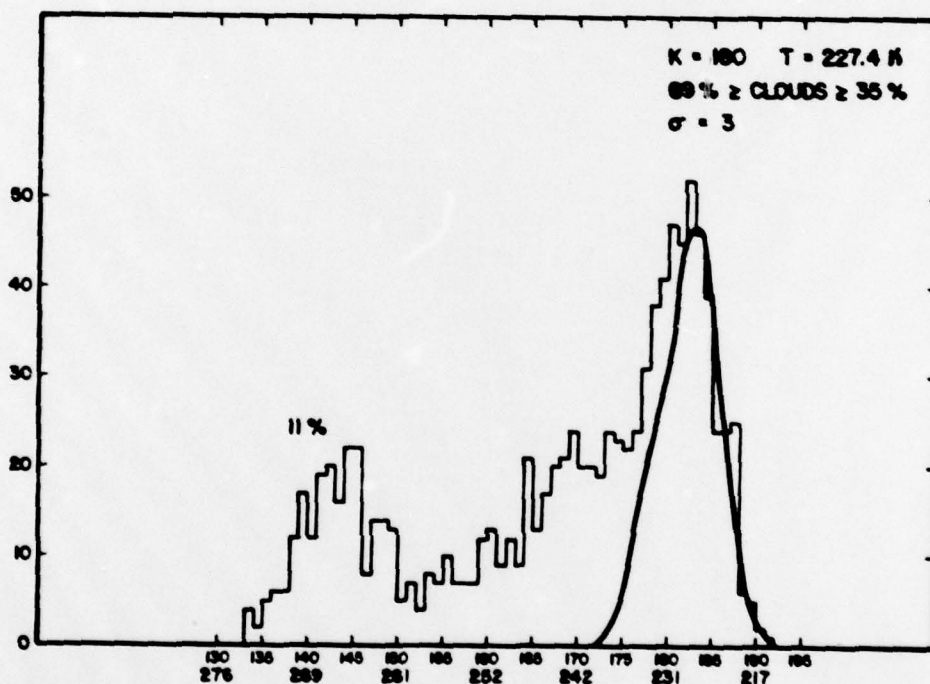


Fig. 1. Histogram of count data from Scanning Radiometer. The procedure has been applied to data from the  $10.5 - 12.5 \mu_m$  infrared channel of the Scanning Radiometer. The technique was applied to data blocks of 1024 measurements. Figs. 1 and 2 are

histograms of such data. The examples are from the Pacific Ocean region for July 12, 1975.

In each figure  $K$  is the truncation point used ( $K = k_m$ ),  $T$  is the temperature estimated for a completely cloud covered f-o-v, the assumed standard deviation of the instrument was three counts, the heavy black curve is the theoretical density (normal distribution) of completely cloud covered f-o-v's. For example, in Fig. 1, the truncation point was 180 counts, the temperature was  $227.4^\circ \text{K}$ , at least 35 per cent of the f-o-v's were covered by the clouds and at least 11 per cent were cloud free (at least of the higher level clouds).

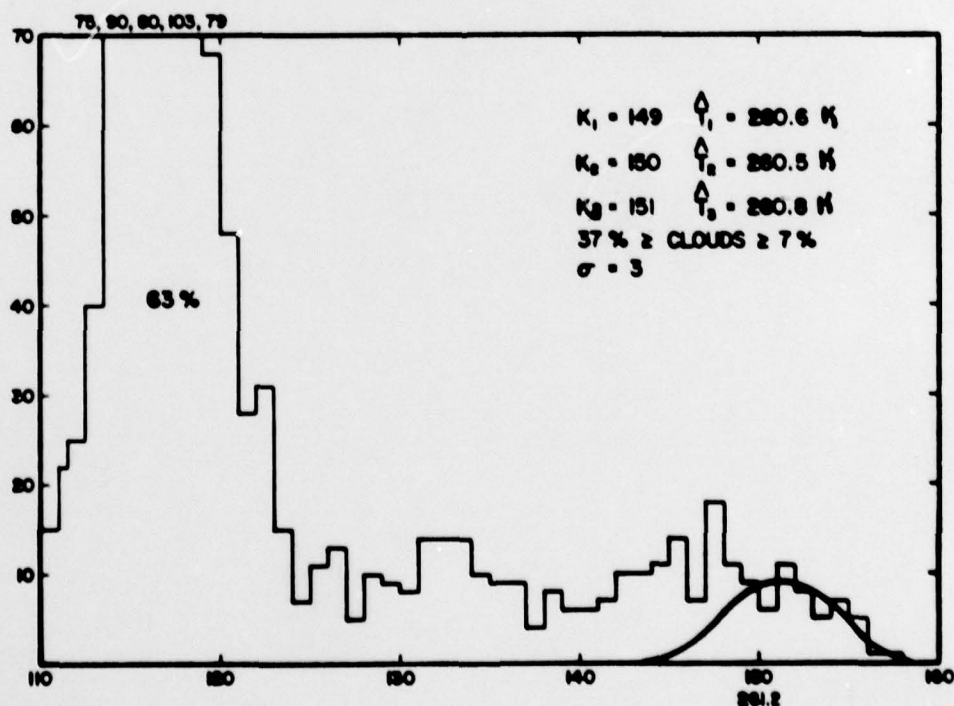


Fig. 2. Histogram of count data from Scanning Radiometer.



In Fig. 2 three different truncation points or  $K$ 's were used. Note the small change in the temperature values.

### CONCLUSIONS

A technique has been presented which gives simple and objective estimates of bounds on certain cloud parameters. In many situations the technique may not give results which will be independently useful. However, they can be used as starting points or checks on situations of the radiative transfer equation. The possible value of the technique comes from the following considerations: with the cloud data given by this technique certain techniques for retrieving atmospheric parameters which have not been practical in the past may now be practical; since the technique is so simple it could be implemented on the satellite, and hence would solve many of the data flow problems associated with small f-o-v instruments.

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